proposed in /9/ for certain polymers for a time-indpendent operator μ° of the form (13).
Therefore, unlike infinitesimal strains, the construction of a viscoelastic analogue of the elastic law is not unique for finite strains.

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A METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS USING DIFFERENTIABLE TRIGONOMETRIC FOURIER SERIES*

A.I. POLUBARINOVA

A method for representing a function of two variables $u(x, y)$, that is defined in the square $\sigma = [0, \pi] \times [0, \pi]$, is presented in the form of a combination of polynomials and differentiable trigonometric series. Such a representation enables problems to be solved in which the unknown function is defined from partial differential equations and has some partial derivatives at the border of the square domain of higher order than the order of the equation. Expansion in a trigonometric series is carried out by a system of functions $\{\sin mx, m = 1, 2, 3 \ldots\}$ that is full in $[0, \pi]$ and in a double series by a system of functions $\{\sin mx \sin ny, m, n = 1, 2, 3, \ldots\}$ that is full in σ. For solving real problems, expansion by such a system of functions can be preferable to expansion by an ordinary trigonometric system of sines and cosines /1, 2/. Using the representation of a function of two variables referred to above the problem of the bending of an anisotropic plate with non-uniform boundary conditions is solved.

1. Formulation and foundation of the method. *Definition 1.* A function $f(x), x \in [0, \pi]$ is even (odd) over $[0, \pi]$ relative to the point $\pi/2$ if $f(x) = f(\pi - x), x \in [0, \pi], (f(x) = -f(\pi - x), x \in [0, \pi])$. A function $F(x, y), (x, y) \in \sigma$ is even or odd in x and y if the corresponding relations on the argument x or y are satisfied.

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Definition 2. A function $F(x, y)$, $(x, y) \in \sigma$ is a function with strictly defined parity if the parity of $F(x, y)$ is known for every argument. This concept carries over analogously for a function of one argument.

We shall formulate a lemma on the possibility of termwise differentiation of the Fourier series of a function $f(x)$ according to a system of functions $\{\sin mx, m = 1, 2, 3, \ldots\}$ on $[0, \pi]$.

Lemma 1. Let $f(x)$ be a function with strictly defined parity. Let there exist in $[0, \pi]$ continuous derivatives f_x^{2l}, $l = 1, 2, \ldots p + 1$, where

$$f_x^{2l}(0) = 0, \quad l = 0, 1, \ldots p \tag{1.1}$$

Moreover, let the derivative f_x^{2p+2} be representable as a Fourier sine series. Then the Fourier sine series of the function $f(x)$ can be differentiated term by term $2p + 2$ times.

Conditions (1.1) ensure the continuity of an odd extension of the function $f(x)$ on $[-\pi, 0]$ and also of the even derivatives of this extension right up to order $2p$. At the same time the derivatives $f_x^{2l+1}, l = 0, 1, \ldots p$ are automatically continuous in $[-\pi, \pi]$ as the functions are even in this interval. The equalities $f_x^k(-\pi) = f_x^k(\pi)$, $k = 0, 1, \ldots 2p + 2$, are also obvious. Thus the well-known conditions /3/ are satisfied for the termwise differentiability of the Fourier series of f right up to order $2p + 2$.

Lemma 2. Let $F(x, y)$ be a function with strictly defined parity. Let there exist everywhere in σ continuous partial derivatives F_{xy}^{2l}, $l = 1, 2, \ldots p + 1$ that satisfy the conditions $F_x^{2l}(0, y) = 0$, $F_y^{2l}(x, 0) = 0$, $l = 0, 1, \ldots p$. Moreover, let all derivatives F_{xy}^{2p+2} be representable in σ by their double Fourier sine series. Then the double Fourier sine series of the function $F(x, y)$ can be differentiated term by term to obtain the derivatives $F_{xy}^{r, s}$, $r + s \leqslant 2p + 2$.

Extending $F(x, y)$ in an odd way (argument by argument in $[-\pi, 0]$ and by reasoning in the same way as for the proof of Lemma 1, we arrive at the known conditions for termwise differentiability of the double Fourier series of $F(x, y)$, $x, y \in [-\pi, \pi] \times [-\pi, \pi]$.

Theorem. Suppose that $u(x, y)$ is a function with strictly defined parity in σ. Suppose that the partial derivatives $u_x^{2p}(0, y)$, $u_y^{2p}(x, 0)$, $u_{xy}^{2p+2}(0, 0)$ exist. Then the unique representation

$$u(x, y) = F(x, y) + \sum_k h_k(x) \psi_k(y) + \sum_c g_c(y) \varphi_c(x) + \sum_{k, c} d_{kc} h_k(x) g_c(y) \tag{1.2}$$

exists, where: 1) $h_k(x)$, $g_c(y)$ are any polynomials in x and y respectively that have exactly the same parity as $u(x, y)$ with respect to the corresponding arguments, and have the following properties: an even polynomial $h_k(x)$ has degree $2k$, an odd polynomial has degree $2k + 1$ and $d^{2k} h_k(0)/dx^{2k} \neq 0$; analogous conditions are applied for $g_c(y)$ depending on its parity;

2) the unknown functions $F(x, y)$, $\varphi_c(x)$, $\psi_k(y)$ have exactly the same parity as $u(x, y)$ with respect to the corresponding arguments and have partial derivatives that satisfy the conditions:

$$F_x^{2l}(0, y) = 0, \quad F_y^{2l}(x, 0) = 0, \quad \frac{d^{2l}\varphi_c(0)}{dx^{2l}} = 0, \quad \frac{d^{2l}\psi_k(0)}{dy^{2l}} = 0 \tag{1.3}$$

Here and afterwards unless otherwise stated the indices c, k, l, q, r, and s take the values $0, 1, \ldots p$; summation over these indices is carried out from 0 to p.

Proof. Uniqueness. For brevity we shall put

$$h_k^{2l}(x) \equiv d^{2l} h_k(x)/dx^{2l}, \quad g_c^{2l}(y) \equiv d^{2l} g_c(y)/dy^{2l}$$

Suppose that there exists a representation (1.2) for $u(x, y)$, and $F(x, y)$, $\varphi_c(x)$, $\psi_k(y)$ satisfy conditions (1.3). We differentiate the equality (1.2) $2l$ times with respect to x for $x = 0$. Taking into account the first and third conditions of (1.3) we obtain

$$\sum_k h_k^{2l}(0) \psi_k(y) = u_x^{2l}(0, y) - \sum_{k, c} d_{kc} h_k^{2l}(0) g_c(y) \tag{1.4}$$

Differentiating (1.2) $2q$ times with respect to y for $y = 0$ and taking into account the first and fourth conditions of (1.3) we obtain

$$\sum_c g_c^{2q}(0) \varphi_c(x) = u_{xy}^{2q}(x, 0) - \sum_{k, c} d_{kc} h_k(x) g_c^{2q}(0) \tag{1.5}$$

Differentiation of (1.4) $2q$ times with respect to y for $y = 0$ taking account of the fourth condition of (1.3) gives

$$\sum_{k, c} d_{kc} h_k^{2l}(0) g_c^{2q}(0) = u_{xy}^{2l, 2q}(0, 0) \tag{1.6}$$

In system (1.6) there are $(p + 1)^2$ equations and the same number of unknowns d_{kc}. We shall show that their solution exists and is unique. The proof is by induction. In (1.6) putting $l = q = p$ we obtain $d_{pp} h_p^{2p}(0) g_p^{2p}(0) = u_{xy}^{2p, 2p}(0, 0)$ since $h_k^{2p} = g_c^{2p} \equiv 0$ for $k, c < p$, and $h_p^{2p}(0) \neq 0$,

$g_p^{2p}(0) \neq 0$ in view of condition 1) of the theorem. From this we can uniquely define d_{pp}.

Suppose that all the d_{kc}, $k + c \geqslant t$ have been found using all the equations of system (1.6) for which $l + q \geqslant t$. We shall find all the d_{rs} for which $r + s = t - 1$. For this we consider all the equations of system (1.6) with numbers $l = r$, $q = s$. Each such equation contains only one unknown d_{rs} with coefficients $h_r^{2r}(0)\, g_s^{2s}(0) \neq 0$ in view of condition 1). The remaining d_{kc} are such that $k + c \geqslant t$, i.e. they are known. Thus from all the equations with numbers $l = r$, $q = s$, $r + s = t - 1$ all the d_{rs} are uniquely defined. The existence and uniqueness of the solution of system (1.6) is proved.

Substituting these uniquely found d_{kc} into (1.4) and (1.5) we obtain two linear systems of equations for finding $\varphi_c(x), \psi_k(y)$. In view of condition 1) these systems will have a zero determinant and, consequently, $\varphi_c(x)$ and $\psi_k(y)$ can be uniquely defined from them. Substituting $\varphi_c(x), \psi_k(y)$ and d_{kc} into (1.2) we obtain $F(x, y)$ uniquely.

Existence. We will prove the existence of the representation (1.2) with the functions $F(x, y)$, $\varphi_c(x), \psi_k(y)$ that satisfy conditions (1.3) d_{kc} is found from system (1.6) and the $\varphi_c(x)$, $\psi_k(y)$ that correspond to the d_{kc} are found using (1.4), (1.5), $F(x, y)$ is found from (1.2). We shall prove that the functions obtained satisfy conditions (1.3). We differentiate (1.4) $2q$ times with respect to y for $y = 0$. On the basis of (1.6) we have

$$\sum_k h_k^{2l}(0)\, \psi_k^{2q}(0) = 0$$

The determinant of this system is non-zero in view of condition 1). This means that the system has only a trivial solution. This proves the fourth condion of (1.3). The third condition of (1.3) is proved analogously.

Differentiating Eq. (1.2) $2l$ times with respect to x for $x = 0$ and using the third condition of (1.3) and Eq. (1.4) we ascertain the correctness of the first condition of (1.3). The second codition is proved analogously.

Corollary. Now let $u(x, y)$ be a function with strictly defined parity that has continuous derivatives u_{xy}^{2p+2} in σ; $u_{xy}^{2p,\, 2p+2}(0, y)$, $u_{xy}^{2p+2,\, 2p}(x, 0)$ on $[0, \pi]$ and let all of them be representable by their Fourier sine series in σ and on $[0, \pi]$ respectively. Then from the theorem and Lemmas 1, 2 we have the existence and uniqueness of the representation

$$u(x, y) = \sum_{m,\, n} F_{mn} \sin mx \sin ny + \sum_k h_k(x) \sum_n \psi_n^k \sin ny + \sum_c g_c(y) \sum_m \varphi_m{}^c \sin mx + \sum_{k,\, c} d_{kc} h_k(x)\, g_c(y) \tag{1.7}$$

where all the series can be differentiated term by term to obtain the partial derivatives $u_{xy}^{r,\, s}$, $r + s \leqslant 2p + 2$; $m = 1, 3, 5, \ldots$, $u(x, y)$ x, $m = 2, 4, 6, \ldots$, if $u(x, y)$ is an odd function in x. Depending on the parity of $u(x, y)$ with respect to y, values of the index n are defined analogously.

2. Calculation of an anisotropic plate.

Consider the boundary value problem for an anisotropic plate in the quadratic domain σ with border Γ°

$$A \frac{\partial^4 w}{\partial x^4} + B \frac{\partial^4 w}{\partial x^2 \partial y^2} + C \frac{\partial^4 w}{\partial y^4} = P^\circ(x, y) \tag{2.1}$$

$$w\,|_{\Gamma^\circ} = \alpha^\circ(x, y), \quad \partial w/\partial n\,|_{\Gamma^\circ} = -\beta^\circ(x, y)$$

The functions $\alpha^\circ(x, y)$ and $\beta^\circ(x, y)$ are given in Γ°. Each of the functions $P^\circ(x, y)$, $\alpha^\circ(x, y)$, $\beta^\circ(x, y)$ is expanded in four terms so that each is a function with strictly defined parity. We shall look for a solution to problem (2.1) in the form of a sum of the solutions of the four problems of the form

$$A \frac{\partial^4 u}{\partial x^4} + B \frac{\partial^4 u}{\partial x^2 \partial y^2} + C \frac{\partial^4 u}{\partial y^4} = P(x, y) \tag{2.2}$$

$$u\,|_\Gamma = \alpha(x, y), \; \partial u/\partial n\,|_\Gamma = -\beta(x, y) \tag{2.3}$$

where $P(x, y)$, $\alpha(x, y)$ and $\beta(x, y)$ are functions with the same strictly defined parity; $\Gamma = \{(x, 0), (0, y), x, y \in [0, \pi]\}$ is a section of the boundary Γ°.

Assume that the functions $P(x, y)$, $\alpha(x, y)$, $\beta(x, y)$ ensure that the conditions that follow from the theorem are satisfied for the solution of (2.2) and (2.3). Then a solution in the form (1.7) can be sought by putting $p = 1$. In this case all the series in (1.7) can be differentiated term by term for finding the partial derivatives $u_{xy}^{r,\, s}$, $r + s \leqslant 4$.

Here and everywhere afterwards in solving (2.2) and (2.3) the values of the indices m and n and also of the functions $h_0(x)$, $h_1(x)$, $g_0(y)$, and $g_1(y)$ are chosen in the following way:

$P(x, y)$ is even in x: $m = 1, 3, 5, \ldots$, $h_0(x) = \pi/4$, $h_1(x) = \pi x(\pi - x)/8$;

$P(x, y)$ is odd in x: $m = 2, 4, 6, \ldots$, $h_0(x) = (\pi - 2x)/4$, $h_1(x) = x(\pi - x)(\pi - 2x)/24$.

The index n is changed in the same way as m but depending on the parity of $P(x, y)$ with respect to y, $g_0(y) = h_0(y)$, $g_1(y) = h_1(y)$. (The coefficients of the Fourier sine series of the functions $h_0(x), h_1(x)$ have the simplest form: $1/m$ and $1/m^3$ respectively, by which the choice of these functions is determined.)

Substitution of the first boundary condition (2.3) into the representation (1.7) leads to the equality

$$u(x, y) = \alpha_*(x, y) + d_{11}h_1(x)g_1(y) + h_1(x)\sum_n \psi_n^1 \sin ny + g_1(y)\sum_m \varphi_m^1 \sin mx + \sum_{m,n} F_{mn} \sin mx \sin ny \tag{2.4}$$

$$\alpha_*(x, y) = \alpha(x, 0)\, g_{0*}(y) + h_{0*}(x)\, \alpha(0, y) - \alpha(0, 0)\, h_{0*}(x)\, g_{0*}(y)$$

$$h_{0*}(x) = h_0(x)/h_0(0), \quad g_{0*}(y) = g_0(y)/g_0(0)$$

Substitution of (2.4) into Eq.(2.2) and the second boundary condition (2.3) gives

$$u(x, y) = \alpha_*(x, y) + d_{11}h_1(x)g_1(y) + \frac{h_1(x)}{A}\sum_n \zeta_n \sin ny + \frac{g_1(y)}{C}\sum_m \eta_m \sin mx + \sum_{m,n}\left(\frac{m\zeta_n + n\eta_m + P_{mn}}{K_{mn}} - \frac{\zeta_n}{Am^3} - \frac{\eta_m}{Cn^3}\right) \times \sin mx \sin ny \tag{2.5}$$

$$d_{11} = \frac{16}{\pi^2 B}\left[P - A\frac{\partial^4\alpha}{\partial x^4} - C\frac{\partial^4\alpha}{\partial y^4}\right]_{x=y=0}, \quad K_{mn} = Am^4 + Bm^2n^2 + Cn^4$$

The coefficients $\zeta_n = A\Psi_n^1$, $\eta_m = C\varphi_m^1$ are the solution of the infinite system of equations

$$\sum_m \frac{m^2}{K_{mn}}\zeta_n + \sum_m \frac{mn\eta_m}{K_{mn}} = \gamma_n - \sum_m \frac{mQ_{mn}}{K_{mn}} \tag{2.6}$$

$$\sum_n \frac{n^2}{K_{mn}}\eta_m + \sum_n \frac{mn\zeta_n}{K_{mn}} = \delta_m - \sum_n \frac{nQ_{mn}}{K_{mn}}$$

where $\gamma_n, \delta_m, Q_{mn}$ are the coefficients of the Fourier sine series corresponding to the following functions:

$$\gamma(y) = \beta(0, y) + \beta(0, 0)\, g_{0*}(y) - [\alpha(0, y) - \alpha(0, 0)\, g_{0*}(y)]\, h_0'(0) - d_{11}h_1'(0)\, g_1(y), \quad \delta(x) = \beta(x, 0) + \beta(0, 0)\, h_{0*}(x) - [\alpha(x, 0) - \alpha(0, 0)\, h_{0*}(x)]\, g_0'(0) - d_{11}h_1(x)\, g_1'(0), \quad Q(x, y) = P(x, y) - A\frac{\partial^4\alpha}{\partial x^4}(x, 0)\, g_{0*}(y) - C\frac{\partial^4\alpha}{\partial y^4}(0, y)\, h_{0*}(x) - d_{11}Bh_0(x)\, g_0(y)$$

Thus formula (2.5) reflects the solution of (2.2) and (2.3) in terms of the known functions $P(x, y), \alpha(x, y), \beta(x, y), h_0(x), h_1(x), g_0(y), g_1(y)$ and the solution of the infinite system (2.6). All the series in (2.6) can be differentiated term by term to find the derivatives $u_{xy}^{r,s}$, $r + s \leqslant 4$. Expanding $h_1(x), g_1(y)$ in a Fourier sine series we obtain the more compact representation

$$u(x, y) = \alpha_*(x, y) + \sum_{m,n}\left(\frac{m\zeta_n + n\eta_m + Q_{mn}}{K_{mn}} + \frac{d_{11}}{m^3n^3}\right)\sin mx \sin ny$$

of a series in which, however, it is not possible to differentiate termwise.

Solving (2.2) and (2.3) for the components of the load and for the boundary conditions of a different strictly defined parity and adding the four solutions obtained we obtain a solution of (2.1). Note that the condition of ellipticity of (2.1) ensures that $K_{mn} \neq 0$ for any m, n.

The question of the existence and uniqueness of the solution and also of methods of solving system (2.6) are of particular interest. We note that in general, it is not possible to reduce system (2.6) to regular form /4/. However, a change of variables

$$\zeta_n = n\bar{\zeta}_n, \quad \eta_m = m\bar{\eta}_m \tag{2.7}$$

transfers (2.6) to an infinite system which can be solved by the method proposed in /4/ for quasi-regular systems. We often meet the case when $A = C$, $P^\circ(x, y) = P^\circ(y, x), \alpha^\circ(x, y) = \alpha^\circ(y, x)$, $\beta^\circ(x, y) = \beta^\circ(y, x)$, and system (2.6) reduces to a regular form. So that it can be solved by the reduction method /4/, it is sufficient that the order of decrease of the coefficients of the Fourier sine series of the functions $P^\circ(x, y)$, $\alpha^\circ(x, y)$, $\beta^\circ(x, y)$ should be not less than $1/(mn)$.

As an example consider the problem of the bending of a rigid weighted isotropic plate under a uniform load in a domain σ. We have $P^\circ(x, y) \equiv P^\circ$; $A = C = 1$; $B = 2$, $\alpha^\circ(x, y) = \beta^\circ(x, y) \equiv 0$. The deflection is expressed in the following way:

$$w(x, y) = P^\circ \sum_{m,n}\left(\frac{m\zeta_n + n\zeta_m}{(m^2 + n^2)^2} - \frac{8}{\pi^2}\frac{1}{m^3n^3}\right)\sin mx \sin ny \tag{2.8}$$

in terms of the solution $\{\zeta_n\}$ of the infinite system

$$\sum_m \frac{m^2}{(m^2+n^2)^2}\zeta_n + \sum_m \frac{mn\zeta_n}{(m^2+n^2)^2} = -\frac{1}{n^3}, \quad m, n = 1, 3, 5, \ldots \tag{2.9}$$

The series (2.8) converges rapidly. It is sufficient to determine the first five values of ζ_n in system (2.9) by the reduction method, and the corresponding partial sum of the series (2.8) gives a good approxomation to the known solution of the problem /5/.

Note that to solve (2.1) a method analogous to that demonstrated in /1/ was chosen since it is more suitable for solving problems with non-homogeneous boundary conditions. Eq.(2.1) can be solved also using two representations of the unknown function /2/ by using in this case a system of sines that is complete in $[0, \pi]$ for the expansion in a Fourier series.

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SOME COMMENTS ON THE BOOK "HYDRODYNAMICS" BY L.D. LANDAU AND E.M. LIFSHITZ*

A.G. KULIKOVSKII

The book in question (Theoretical physics, 1986, Vol.6) is the third edition of the part of the book "Mechanics of Continuous Media" (1953, 2nd ed.), which is concerned with hydrodynamics and contains some addenda and changes.

Some references are added in the new edition. To establish priorities, the first publications are indicated, often in sources inaccessible to Soviet readers. In view of this, we shall note some mistakes in these references.

1. It is said on p.674 that it will be shown in Sect.130 that, in some special cases, detonation must inevitably correspond to the Chapman-Jouguet condition, and a reference is given, according to which the proof of this condition was obtained by Ya. B. Zel'dovich (1940) /1/, and independently, in some later works. Yet the arguments quoted in Sect.130 (p.678) amount to proving the impossibility of realizing supercompression of detonation $(D < u_2 + a_2)$ in many flows when a rarefaction wave is present behind the detonation wave.

It must be said here that the unrealizability of supercompressed detonation in these flows had been well established before 1940. This was mentioned directly in /1/ with references to Wendlant's work /2/ (1924) and to Jost's book /3/ (1939), to which there are no references in the present book. There are also no references to A.A. Grib's important results, published in his Candidate Dissertation presented in 1939 and defended in February 1940 (see also /4/) and cited by many authors (see e.g., /5/). In /6/ there is a reference to A.A. Grib in connection with the study of detonation, though not to /4/, but to a different paper not relating to detonation.

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